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Frames and fermionic excitations of vortices in superfluid helium

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Abstract. In this paper I consider vortical structures in superfluid helium in the form of knots and links. Then I give some basic notions and facts from knot theory, necessary to understand the rest of the paper. I consider framed knotted vortices and identify these frames with some excitations. The statistics of these excitations is actually fermionic.

1. Introduction

As is well known, quantum vortices in superfluid helium are treated either as open lines with their ends terminating on free surfaces or walls of the container, or as closed curves. It is a paradigm to consider vortices of this second kind as objects topologically equivalent to circles. In my opinion, this common belief should be revised. Namely, I think that closed vortices in the form of non-trivial knots should be investigated. Additionally, systems of knots in the form of non-trivial links should be taken into consideration.

This is the first of a series of papers, where I examine the consequences of non-trivial topology of vortices for the physics of superfluid helium. These are manifold. In this paper I discuss fermionic excitations of knotted vortices. In the next paper I will argue that fermionic excitations are also possible in bulk superfluid filled with knotted vortex lines (Owczarek 1993a) and then I will investigate their role in the theory of λ phase transition (Owczarek 1993b). I will not touch problems connected with superfluid turbulence. However, the very existence of such structures as knotted and linked vortex lines in the turbulent phase seems to be rather obvious (Schwartz 1985) and should be a subject of investigation in the near future.

The plan of this paper is as follows. Firstly, I will introduce some basic notions and results from the knot theory to convince readers, to establish notation and to present some useful facts. Secondly, I will show how the excitations of the vortices can originate. Finally, I will briefly discuss experimental results of the experiment (Ohbayashi *et al* 1990) from the point of view of the new approach proposed in this paper.

2. Basic notions from the knot theory

Knot theory is the theory of embeddings of a circle into three-dimensional space. Such embeddings are called knots. Originally it was only a three-dimensional sphere, i.e. compactified three-dimensional Euclidean space, where these knots were placed. The problem of the classification of knots has not been fully resolved until now, but the idea is to build up some mathematical objects characterizing knots, invariant under transformations changing only their shape and not their topology. Many such invariants are known, but none of them characterizes knots uniquely. Recently significant progress has been made in the theory. Firstly, Jones (1985) constructed such invariants, now known as Jones polynomials. These are much more effective in differentiating knots than invariants known previously, and can be applied to links of knots, i.e. linked structures obtained by embedding at least two circles in a three-dimensional space. Moreover, their construction has some connections with two-dimensional exactly solved models of statistical physics. The second big step forward was made by Witten (1989). He described Jones polynomials in an entirely three-dimensional way in the framework of a topological quantum gauge field theory. In my considerations of knotted vortices I apply the Witten approach widely. Therefore I will briefly present the main points of it now.

Knot invariants in this approach are constructed in the framework of the Chern–Simons gauge field theory. Such a theory could be formulated over any three-dimensional manifold M using any gauge field A for arbitrary compact (simple or Abelian) gauge group G.

The Chern–Simons action for the gauge field:

$$S(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}\,(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A) \tag{1}$$

where Tr is the appropriately renormed Killing metric for the group G (for an Abelian group it is any appropriately renormed metric) and k is an arbitrary integer number. This action is invariant under gauge transformations, or rather $\exp[\mathrm{i}S(A)]$ is. As one can see, it is built without the use of any metric on the manifold M. As a result, the quantities:

$$Z\Big(M, \bigcup_{i=1}^{n} C_i\Big) = \int DA\ \exp[\mathrm{i}S(A)] \prod_{i=1}^{n} \mathrm{Tr}_{R_i}\Big[P\exp\Big(\oint_{C_i} A\Big)\Big] \tag{2}$$

where R_i are some representations of G, assigned to knots C_i in M, P stands for an orientation of C_i, are also topological invariants.

When there are no knots in the manifold M, this is a topological invariant of the manifold itself. On the other hand, when $M = S^3$ these invariants are directly connected with Jones polynomials as Witten has shown. The case of one knot is interesting because in this case one should regularize the invariant in some way. The regularization scheme proposed by Witten resembles a physical *splitting point procedure*. He introduces *frames* of knots, i.e. fields of normal vectors. One can then consider an image C' of the original knot C under infinitesimal translation along normal vector field. The invariant calculated for the pair C, C' is the regularized invariant for C. Its value depends on the frame and changes under changing of a frame by a multiple of an integer number i.e. by a well defined amount. The role of these frames is crucial in my further consideration of knotted vortices in this paper.

This general scheme has its rather simple counterpart in the case of the gauge group G being $U(1)$. This gauge group is very important in the theory of superfluidity (Kleinert 1982, Peradzyński 1990, Owczarek 1991). Representations of this gauge group are very simple, one-dimensional, and are characterized by integers n_i:

$$R_{n_i} : U(1) \ni z \mapsto z^{n_i} \in U(1) \subset C^*. \tag{3}$$

The Chern–Simons action in this case is

$$S(A) = \frac{k}{4\pi}\int_M A \wedge \mathrm{d}A \tag{4}$$

and the invariants for $M = S^3$ are

$$Z\Big(M, \bigcup_{i=1}^{n} C_i\Big) = \exp\Big(\frac{1}{2k}\sum_{a,b} n_a n_b \Phi(C_a, C_b)\Big) \tag{5}$$

where

$$\Phi(C_a, C_b) = \frac{1}{4\pi}\int_{C_a} dx^i \int_{C_b} dy^j \varepsilon^{ijk} \frac{(x-y)^k}{|x-y|^3} \tag{6}$$

with n_a and n_b integers characterizing representation to C_a and C_b respectively, is the Gauss linking number, a well known topological invariant.

3. Application to superfluid helium

In the hydrodynamics of an ideal fluid there is an invariant of motion called helicity

$$H = \int_{R^3} d^3x \, \overline{v} \, \mathrm{rot} \, \overline{v} \tag{7}$$

where $\overline{v}$ stands for velocity field of the fluid.

For vorticity concentrated on very thin vortex lines this invariant is proportional to the Gauss linking number of these lines (Moffatt 1969). Therefore it is a topological invariant and has a good interpretation in the framework of the knot theory, as can be seen from the formula (6) above.

Peradzyński (1990) considered the same invariant in the case of superfluid helium in the framework of an explicitly $U(1)$-invariant description of superfluid helium. He constructed an action 1-form

$$A = \overline{v}_s \, d\overline{x} - \left(\mu + \tfrac{1}{2}\overline{v}_s^2\right) dt \tag{8}$$

where $\overline{v}_s$ is the superfluid velocity field, and μ is the chemical potential. This action has its relativistic counterpart

$$A = v_{s_\mu} \, dx^\mu \tag{9}$$

where v_{s_μ} is the superfluid four-velocity field.

The quantity

$$H = \int A \wedge dA \tag{10}$$

is an invariant of motion, a superfluid counterpart of helicity. Then one has explicitly the $U(1)$-invariant equation of motion

$$dA = J \tag{11}$$

(where J is the spacelike vorticity 2-form) being fulfilled.

Of course, the helicity invariant is proportional to the Gauss linking number as in the case of an ideal fluid.

On the other hand, the phenomenon of reconnection of vortex lines could change locally the helicity and the linking number in ordinary fluids (Moffatt 1969) and in superfluids (Schwartz 1985) as well. Such effects are caused by dissipative processes in fluids and could be responsible for such phenomena as superfluid turbulence (Schwartz 1985) and λ-phase transition (Owczarek 1993b).

How to interpret frames of knots in the theory of superfluidity is an interesting problem. As we have seen, frames of knots schould be understood as normal vector fields for vortices. There are many examples of applications of normal vector fields in the theory of quantum vortices (see for example Owczarek and Słupski 1992). These fields can be connected with excitations of vortices. In my opinion, frames of vortices are actually very special excitations, which are quantized in a way, because they could be numbered by integers as one can see from the formula describing a change of the invariant under a change of frame.

There is possibly a connection between a scheme of geometric quantization of vortices and these excitations. There are similar suggestions in the article by Penna and Spera (1989), but this has some weak points (Słupski 1991). Work on these problems is in progress.

4. Frames and fermionic excitations of vortices

There are two possible physical interpretations of knotted curves in three-dimensional space. They depend on the signature of the metric tensor that is finally introduced in the target space for embeddings.

If the signature of the metric is Euclidean, we can interpret the three-dimensional space as a t = constant hypersurface in four-dimensional space–time. When the topology of the system of knotted lines is conserved in time, we can calculate our invariants at an instant of time and this is the case considered.

On the other hand, if the signature of the three-dimensional space is Lorentzian, the interpretation is different. Our three-dimensional space is similar to the normal four-dimensional space–time with the only difference being that the number of space-like dimensions is two.

Lines in this space-time can be interpreted as trajectories of particles (especially when there are representations of the group G assigned to them and such representations could be connected with charges of the particles). Such situations are known in physics, for example in high-temperature superconductivity where the motion of an electron is limited to a two-dimensional plane. There is also another possibility. We can consider projections of trajectories on a two-dimensional plane. This procedure leads to the reduction of the dimensionality of space-time from 3+1 to 2+1. Such an interpretation seems to be natural in the theory of superfluidity (Rasetti and Regge 1975). These authors showed that the dynamics of vortices in superfluid helium is well described by the dynamics of their projections on a two-dimensional plane. In the vicinity of the λ point we can ignore the influence of superfluid helium constituents other than quantum vortices. As a result, its effective dynamics is $(2+1)$-dimensional and is apparently connected with knot theory, where two-dimensional projections of knots satisfactorily characterize them (figure 1). I should only mention that Rasetti and Regge investigated non-gauged vortices and I have no evidence that their conclusions are also true in the case of gauged vortices.

Both of these interpretations could be useful in the theory of vortices in a superfluid. We could use the two approaches because of the following features of the helicity invariant seen from Peradzyński's (1990) paper: A is the 1-form in the four-dimensional space–time (relativistic or not). Because the equation of motion is fulfilled

$$\mathrm{d}A = J \tag{12}$$

where J is the spacelike 2-form, we have

$$\mathrm{d}(A \wedge \mathrm{d}A) = \mathrm{d}A \wedge \mathrm{d}A = J \wedge J = 0 \tag{13}$$

as a 4-form built with the use of three base 1-forms $\mathrm{d}x^i$, $i = 1, 2, 3$.

Then

$$\int_{\Sigma_1} A \wedge \mathrm{d}A - \int_{\Sigma_0} A \wedge \mathrm{d}A = \int_V \mathrm{d}(A \wedge \mathrm{d}A) = 0 \tag{14}$$

for an arbitrary pair of hypersurfaces Σ_0 and Σ_1, which are boundaries of an area V of space-time. These hypersurfaces could be space-like or time-like; it does not matter. This insensitivity of the result to the character of the hypersurface is the cause of the appearance of various kinds of approach to knotted vortices in a superfluid.

In my further considerations in this paper I will apply the Lorentzian point of view. In this case knotted lines are formally identified with trajectories of particles. Such particles can change their statistics from bosonic to fermionic or vice versa. I use Polyakov's (1989) results to show this.

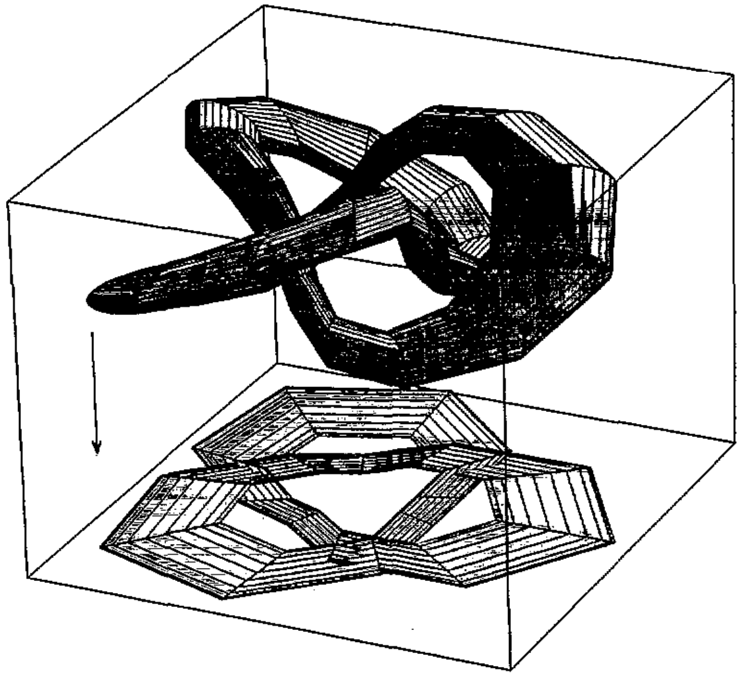

Figure 1. Dimensional reduction for knots as a projection operation.

When one considers a statistical sum over trajectories of particles without any internal structure it is usually written as

$$Z = \sum_{P} \exp[-mL(P)] \tag{15}$$

where $\sum$ is over all possible trajectories P (in our case they are all closed) and m is a characteristic mass scale.

Polyakov proposes to treat the knots as trajectories of particles dressed by the gauge field A, which in his approach is already a $U(1)$ gauge field. Then the statistical sum is written in the form

$$Z = \sum_{P} \exp(-mL)\left\langle \exp\left(\mathrm{i} \oint_{P} A \right) \right\rangle \tag{16}$$

where

$$\left\langle \exp\left(\mathrm{i} \oint_{P} A \right) \right\rangle = \int DA \exp[\mathrm{i}S(A)] \exp\left(\mathrm{i} \oint_{P} A \right) \tag{17}$$

and $S(A)$ is the Chern–Simons action. Therefore $\langle \exp(\mathrm{i} \oint_P A) \rangle$ is the very same invariant as considered by Witten, except for the unimportant multiplication by i. Polyakov regularizes it in a slightly different way from Witten. The result is

$$\left\langle \exp\left(\mathrm{i} \oint_{P} A \right) \right\rangle = \exp\left(\mathrm{i} \int_{0}^{L} C(s)\, \mathrm{d}s \right) \tag{18}$$

where s is a parameter along the curve P and $C(s)$ is the torsion of P for the value of the parameter equal to s.

As Polyakov (1989) has shown, the propagator of the particle in the momentum representation is equal to

$$G(\overline{p}) = 1/(m - \mathrm{i}\overline{\sigma}\,\overline{p}) \tag{19}$$

where $\overline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices.

This quantity is the propagator of a particle of spin $\frac{1}{2}$ i.e. a fermion, with mass equal to m.

We know that m is a characteristic mass scale, which is inversely proportional to a characteristic length scale of the theory. Such a characteristic length scale for superfluid helium can be identified with the correlation length. Usually such a correlation length in superfluid helium is of the order of 1 Å and this corresponds to rather heavy particles, which are difficult to observe. The only way to observe such excitations is to perform experiments for temperatures very close to the λ point.

It seems that there already exists an experiment (Ohbayashi *et al* 1990), the results of which seem to be unclear to the authors and could possibly be explained in the framework of my approach. In the experiment Raman scattering of light by a superfluid was investigated. A difference occurred between the experimental results and the predictions of the theory of Raman scattering. These divergences exist near the λ point. In my opinion, we can understand this difference if we accept the point of view that light could be scattered by frames of vortices, appearing near the λ-point temperature.

5. Conclusions

As we have seen, frames of knotted vortices can be interpreted as their specific excitations. Their interesting feature is that being dressed by the $U(1)$ gauge field A, they behave, from a statistical point of view, as fermions.

Moreover, I believe that such fermionic excitations can cause some interesting effects. In my future paper (Owczarek 1993b) I will discuss the role of fermionic excitations in a bulk superfluid on the λ phase transition.

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